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On a new momentum expression

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Abstract. We discuss some motivations that lead to the proposal of a generalized 4-momentum, which takes the form $p^\alpha = (1/c^2)(u_\nu p^\nu)u^\alpha + \kappa\, \mathrm{d}u^\alpha/\mathrm{d}\tau$. This expression has been applied recently to the problem of the Lorentz–Dirac equation and yields well behaved solutions. Here we apply it to a free particle and discuss some possible consequences.

1. Introduction

In classical mechanics, momentum and velocity are defined to be directly proportional to each other as given by

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{1.1}$$

All of classical mechanics rests upon this simple relationship. The validity of (1.1) was considered a paradigm, until the advent of the theory of relativity, which replaces (1.1) by

$$p^\alpha = mu^\alpha. \tag{1.2}$$

This covariant generalization of (1.1) becomes the foundation of relativistic mechanics. Its validity is again considered to be universal, at least for a free particle. However, there is a simple but important difference between (1.1) and (1.2). Although there is no restriction imposed on the $\boldsymbol{v}$, there is a condition

$$u_\nu u^\nu = c^2 \tag{1.3}$$

that u^α must always satisfy.

There now arises the question of whether (1.2) and (1.3) are compatible under all conditions? The answer depends on how p^α is interpreted. One interpretation is that p^α does not possess any independent physical meaning originally. Its physical meaning is acquired precisely from (1.2). In this case, equation (1.2) is strictly a matter of defining p^α by u^α. As such, equations (1.2) and (1.3) can always be made to agree with each other. An alternative interpretation is that p^α possesses independent physical meaning: it is this quantity that will respond to the external physical conditions and communicate this response to modify the state of the motion of the particle that carries it. In this case, equation (1.2) is a kind of constitutive relationship which is valid only under certain physical conditions, indicating that the physical quantity p^α and the spacetime geometrical quantity u^α are not completely independent of each other. If one treats (1.2) as a constitutive relationship rather than just a definition, then one may look for a generalization of (1.2) so that the generalized relationship will manifestly warrant

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the validity of (1.3) and reduce to (1.2) in some way, just as (1.2) will reduce to (1.1) for a small value of velocity. We expect that (1.2) should be a very good approximation for a free particle with large m or small acceleration just as (1.1) is a very good approximation for small $\boldsymbol{v}$. It is for very small m during or shortly after collision, suffering a large acceleration, that (1.2) may become questionable. If we look back at the classical expression (1.1) we find that it is valid for moderate values of m and $\boldsymbol{v}$. Since it must be modified to account for effects at high velocities, it probably will require modification from quantum effects as the small-mass limit is approached. Quantum effects here means microscopic, intrinsic, uncontrollable interactions among some dynamical variables. By treating u^α and p^α as independent dynamical variables whose constitutive relationship is to be found, one may imagine a certain intrinsic interaction between u^α and p^α. It is expected that such a generalized momentum–velocity relationship will approach the established (1.2) asymptotically in time or fluctuate around it, if a particle is isolated in a force-free space.

Following this line of reasoning, we are led to propose a new relationship among momentum, velocity and acceleration.

2. Intrinsic interaction between momentum and velocity

Flat Minkowski systems will be taken as the reference inertial ones. In such systems a spacetime point is denoted by $\{x^\alpha | x^0 = ct, x^1 = x, x^2 = y, x^3 = z\}$. The metric is defined to be $g^{00} = g_{00} = 1$, $g^{ii} = g_{ii} = -1$, for $i = 1, 2, 3$ and $g^{\alpha\beta} = g_{\alpha\beta} = 0$, for $\alpha \neq \beta$. The element of world distance is then given by

$$\mathrm{d}s^2 = (\mathrm{d}x_\alpha)(\mathrm{d}x^\alpha) = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2. \tag{2.1}$$

Defining the proper time to be

$$\mathrm{d}\tau = \frac{1}{c}\sqrt{\mathrm{d}s^2} \tag{2.2}$$

one has

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \tag{2.3}$$

as the definition of 4-velocity. From this definition, 4-velocity must always satisfy

$$u_\nu u^\nu = c^2. \tag{2.4}$$

Condition (2.4) is a direct and sole consequence of the spacetime structure and is independent of the dynamical environment, as long as it does not alter the spacetime structure. The gravitational field shall be excluded from our consideration here. Taking the time derivative of (2.4) one has

$$u_\nu \frac{\mathrm{d}u^\nu}{\mathrm{d}\tau} = u^\nu \frac{\mathrm{d}u_\nu}{\mathrm{d}\tau} = 0. \tag{2.5}$$

A particular solution to (2.5) is

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = \frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = 0. \tag{2.6}$$

All inertial systems satisfy (2.6) and are characterized by it.

The general motion of a particle clearly does not satisfy (2.6). In a force-free space, the motion of a particle can have (2.6) as a solution. In fact, equation (2.6) is conventionally regarded as the sufficient and necessary condition that the motion of a particle is in a force-free space. Here, we depart from this view and consider (2.6) as only a sufficient, but *not a necessary* condition that the motion of a particle is indeed in a force-free space. A force-free space is defined here as that where there exists no external force agent to act on and change the dynamical state of the particle moving in this space.

We shall characterize a dynamical state of the motion by a dynamical quantity called the energy–momentum 4-vector, to be introduced presently, jointly with the velocity 4-vector u^α. We assume that a physical particle possesses and carries along with it some physical quantities that can only be changed by external forces. In their absence such physical quantities will be constants of the motion.

We therefore postulate that every massive physical particle possesses and carries along with its motion a timelike 4-vector called the energy–momentum 4-vector and denoted by $\{p^\alpha | p^0 = E/c, p^1 = p_x, p^2 = p_y, p^3 = p_z\}$. This energy–momentum 4-vector, being timelike, satisfies the condition

$$p_\nu p^\nu = (mc)^2 \tag{2.7}$$

where m is the rest mass. According to Newton's law of motion, the temporal evolution of p^α and the external force 4-vector f^α are related by

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = f^\alpha. \tag{2.8}$$

In the absence of an external force, equation (2.8) is reduced to

$$\frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} = 0. \tag{2.9}$$

We regard condition (2.9) as the sufficient and necessary condition to characterize that the motion of a particle is in a force-free space.

In order that (2.8) or (2.9) can be physically meaningful and mathematically solvable, a constitutive relationship between p^α and u^α must be provided. Conventionally this relationship is given by (1.2). With (1.2), conditions (2.6) and (2.9) become completely equivalent, which specify the motion to be in a force-free space. We shall now depart from the established relationship (1.2).

We require u^α to always satisfy (2.5), even in the force-free case. There are many possibilities for this to occur. Obviously, equation (1.2) satisfies (2.5). However, equation (1.2) is so rigid that momentum and velocity do not have independent meaning. The simplest possibility beyond (1.2) that allows independent meaning shall be considered. It can be easily seen that the equation

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = T^{\alpha\beta} u_\beta \tag{2.10}$$

satisfies (2.5). Here $T^{\alpha\beta} = -T^{\beta\alpha}$ is an antisymmetric tensor. We require that (2.10) involves only intrinsic variables, which in this case are u^α and p^α. Therefore, $T^{\alpha\beta}$ must be constructed from u^α and p^α. The simplest possible $T^{\alpha\beta}$ is

$$T^{\alpha\beta} = \frac{k}{\overline{h}} \left(p^\alpha u^\beta - p^\beta u^\alpha \right) \tag{2.11}$$

where $\overline{h}$ is introduced to make the physical dimensions of the two sides of (2.10) agree. k is a purely dimensionless factor yet to be determined. When (2.11) is substituted into (2.10), the equation

$$p^\alpha = \frac{1}{c^2}(u_\nu p^\nu) u^\alpha + \frac{\overline{h}}{kc^2} \frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} \tag{2.12}$$

is obtained. The new momentum–velocity relationship (2.12) is considered to be the constitutive relationship among p^α, u^α and $\mathrm{d}u^\alpha/\mathrm{d}\tau$ for a particle, independent of whatever dynamical environment the particle is situated in.

This new expression (2.12) has been applied to the problem of the Lorentz–Dirac equation, yielding well-behaved solutions [1]. Here we shall apply it to the simpler problem of a non-radiating free particle.

3. Motion of a free particle

We must now test whether this new relationship provides a reasonable extension to the established relations (1.1) and (1.2). For simplicity, only the force-free situation, where p^α satisfies (2.9) rather than (2.8) shall be considered here. Remembering that p^α satisfies (2.9), we obtain from (2.12)

$$\begin{aligned} p_\nu p^\nu = (mc)^2 &= \frac{1}{c^2}(u_\nu p^\nu)^2 + \frac{\hbar}{kc^2}\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu p^\nu) \\ &= \frac{1}{c^2}(u_\nu p^\nu)^2 + \left(\frac{\hbar}{kc^2}\right)^2\left(\frac{\mathrm{d}u_\nu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right). \end{aligned} \tag{3.1}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(u_\nu p^\nu) = \frac{\hbar}{kc^2}\left(\frac{\mathrm{d}u_\nu}{\mathrm{d}\tau}\right)\left(\frac{\mathrm{d}u^\nu}{\mathrm{d}\tau}\right) \leqslant 0 \tag{3.2}$$

and the general solution to (3.1) is given by

$$(u_\nu p^\nu) = mc^2\frac{\left[\exp\left(kmc^2\tau/\hbar\right) + \eta\exp\left(-kmc^2\tau/\hbar\right)\right]}{\left[\exp\left(kmc^2\tau/\hbar\right) - \eta\exp\left(-kmc^2\tau/\hbar\right)\right]} \tag{3.3}$$

where η is an integration constant. In order to have p^α parallel rather than antiparallel to u^α asymptotically, $(u_\nu p^\nu)$ is constrained to be positive. Hence, from (3.3), $k > 0$. In order to satisfy (3.2) for all time, η must be greater than zero; in order that the expression (3.3) may not become infinity at some positive finite time η must be confined to be less than 1. Now, substituting (3.3) into (2.12), a linear equation with coefficients depending on τ is obtained from the apparently nonlinear equation. Its solution is

$$u^\alpha = \frac{p^\alpha}{(mc)^2}(u_\nu p^\nu) + q^\alpha\left[\exp\left(\frac{kmc^2\tau}{\hbar}\right) - \eta\exp\left(\frac{-kmc^2\tau}{\hbar}\right)\right]^{-1} \tag{3.4}$$

where q^α are integration constants. Since (3.4) must always satisfy (2.4), the integration constants must be related by

$$q_\nu q^\nu = -4\eta c^2 \tag{3.5}$$

$$q_\nu p^\nu = 0. \tag{3.6}$$

When $\tau = 0$ in (3.4),

$$u^\alpha(0) = \frac{p^\alpha}{m}\frac{1+\eta}{1-\eta} + q^\alpha\frac{1}{1-\eta}. \tag{3.7}$$

From (3.5)–(3.7) the integration constants can be determined in terms of the initial conditions yielding the following results:

$$\eta = \frac{[u_\nu(0)p^\nu - mc^2]}{[u_\nu(0)p^\nu + mc^2]} \tag{3.8}$$

$$q^\alpha = (1-\eta)u^\alpha(0) - (1+\eta)\frac{p^\alpha}{m}. \tag{3.9}$$

Therefore, equation (2.12) is completely solved. From (3.3) and (3.4), it follows that

$$\lim_{\tau\to\infty} u^\alpha(\tau) = \frac{p^\alpha}{m}. \tag{3.10}$$

Thus, a particle left in a force-free space will eventually tend to obey the usual relationship (1.2). We have already noted this asymptotic tendency in connection with the Lorentz–Dirac equation [1]. However, in that case, such alignment may be suspected to be helped by the radiation reaction. Here we see that it is an intrinsic built-in tendency due to the interaction between u^α and p^α.

Now, there is one interesting thing to be observed. Our ordinary concept about inertial mass is that it is a characterization of a particle to resist the change imposed by the external agents. The larger the mass the lesser the change. Here we see a second role. The larger the mass the easier for the momentum to align the velocity to be parallel to it, as can be seen from (3.3) and (3.4).

For large masses the approach to the limit (3.10) is so rapid that (2.12) gives practically the same result as the usual (1.2). Only, when the first and the second terms in (2.12) are comparable will (2.12) deviate significantly from (1.2). Due to the smallness of the factor $\hbar/c^2$, in order for this situation to occur, either $km \ll 1$, $|u^\alpha| \ll |\mathrm{d}u^\alpha/\mathrm{d}\tau|$, or a combination of them must obtain. Assuming that $k \approx 1$ and $u^\alpha \approx \mathrm{d}u^\alpha/\mathrm{d}\tau$, then even for an electron, which has the smallest known mass, the two terms in (2.12) are still not of the same order of magnitude. Thus, as long as the value of k is not too small, equations (2.12) and (1.2) can be considered as the same for practical purposes. Nevertheless, equations (2.12) and (1.2) are conceptually and structurally different. In (1.2), the concept of momentum is derived from that of the velocity; in (2.12) the concept of momentum has its own independent meaning. In (1.2), any change in the momentum must be completely absorbed by the velocity alone; in (2.12), such changes can be shared by the velocity and the acceleration. This possibility of properly sharing the changes of the momentum, between the velocity and the acceleration, may have contributed to make a radiating charged particle avoid running away [1].

4. Generation of spin

According to pre-relativistic classical mechanics, it is difficult to define spin for a point particle. However, in relativistic kinematics, a point particle will acquire the Thomas precession [2, 3] as long as its acceleration and velocity are not parallel to each other. Thomas precession is a strict consequence of the theory of relativity and the peculiar properties of the Lorentz transformation regardless of any specific dynamics of the problem. Thus, it seems to be reasonable to expect that a point particle will acquire spin angular momentum parallel to the Thomas precession. We shall now examine this possibility.

In non-relativistic classical mechanics, orbital angular momentum of a moving particle is defined with respect to the origin of the laboratory system as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{4.1}$$

The covariant relativistic generalization is

$$L^{\alpha\beta} = x^\alpha p^\beta - x^\beta p^\alpha. \tag{4.2}$$

From (3.4) we obtain by integration and differentiation

$$x^\alpha = r^\alpha + \frac{\hbar p^\alpha}{k(mc)^2}\ln R - q^\alpha Q \tag{4.3}$$

$$R = \left[\exp\left(\frac{kmc^2\tau}{\hbar}\right) - \eta\exp\left(\frac{-kmc^2\tau}{\hbar}\right)\right] \tag{4.4}$$

$$Q = \frac{\hbar}{2\sqrt{\eta}kmc^2}\ln\left[\frac{\exp\left(kmc^2\tau/\hbar\right) + \sqrt{\eta}}{\exp\left(kmc^2\tau/\hbar\right) - \sqrt{\eta}}\right] \tag{4.5}$$

$$\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} = -\frac{kmc^2}{\hbar}R^{-2}\left[\frac{4\eta p^\alpha}{m} + q^\alpha\left[\exp\left(\frac{kmc^2\tau}{\hbar}\right) + \eta\exp\left(\frac{-kmc^2\tau}{\hbar}\right)\right]\right] \tag{4.6}$$

where r^α is an integration constant. Thus, the covariant orbital angular momentum for a particle with the new momentum expression in a force-free space is

$$L^{\alpha\beta} = r^\alpha p^\beta - r^\beta p^\alpha - (q^\alpha p^\beta - q^\beta p^\alpha)Q. \tag{4.7}$$

This is not a constant of the motion, as Q is a function of time. However, in a force-free space one expects total angular momentum to be a constant of the motion. Therefore, the variation of the orbital angular momentum must be converted into something we may call spin. If we denote this spin by $S^{\alpha\beta}$ and assuming that at the initial moment it is zero, then the spin at any time afterwards is given by

$$S^{\alpha\beta} = -(q^\alpha p^\beta - q^\beta p^\alpha)[Q(0) - Q(\tau)]. \tag{4.8}$$

As $\tau \to \infty$ a net spin is generated. The Thomas precession is proportional to $u^\alpha(\mathrm{d}u^\beta/\mathrm{d}\tau) - u^\beta(\mathrm{d}u^\alpha/\mathrm{d}\tau)$. From (3.4) and (4.4) we have

$$(q^\alpha p^\beta - q^\beta p^\alpha) = \frac{\hbar}{kc^2}R\left[u^\alpha\frac{\mathrm{d}u^\beta}{\mathrm{d}\tau} - u^\beta\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau}\right]. \tag{4.9}$$

Therefore, the spin generated is parallel to the Thomas precession as expected.

5. Remark

Although expression (2.12) as compared to expression (1.2) can be regarded as new, the idea that velocity and momentum are not always parallel to each other is not a new one. Dirac in his investigation that established the Lorentz–Dirac equation already proposed that for a radiating electron momentum can be any vector function of velocity and acceleration as long as it satisfies the relation $u_\alpha(\mathrm{d}p^\alpha/\mathrm{d}\tau) = 0$. He decided to stay with the usual expression (1.2) by saying that other choices are more complicated than this one and would not be expected to apply to a simple thing like an electron [4]. However, an electron as a radiating and spinning particle does not seem to be that simple a thing. In fact, based on his relativistic quantum equation for the electron Dirac hinted that a new kind of dynamics seems to be implied for the electron [5]. More recently, Barut discussed a possible dynamical system for a radiating electron based on the Lorentz–Dirac equation [6]. He proposed a momentum expression similar to our (2.12) as a mathematical trick of transformation of variables. His expression is not compatible with the above-mentioned condition imposed by Dirac. The idea of Barut seems to be that the spin of the electron comes from the radiation reaction. This idea can be traced to the earlier proposal of Wessel [7]. In a series of papers Wessel constructed an alternative equation to the Dirac equation based on the idea that the radiation reaction is the cause that generated the spin. However, non-radiating particles are known to also carry spin. Thus to base the generation of spin on the radiation reaction seems to be questionable. Whatever the cause that generates the spin, its existence suggests that the momentum and velocity may not be parallel. Theories based on this observation are well discussed in the book by Corben and the references cited therein [8]. Some possible experimental verification of the non-collinearity between velocity and momentum is also discussed [9].

Here we attribute momentum as an intrinsic independent dynamical entity carried by all particles. It is through the interaction of momentum with the external force that the state of motion will be modified in such a way as to always warrant the validity of the universal condition (1.3). Some internal mechanism must then exist for the momentum to communicate its change, due to the external conditions, to the particle as to how velocity and acceleration should be modified. Our expression (2.12) is an initial attempt to include such intrinsic interactions in the simplest way. Although we have not yet found a completely satisfactory way to definitely fix the constant k that appears in the expression (2.12), simple preliminary applications of (2.12) seem to indicate that (2.12) deserves to be investigated further.

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